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A NEW DETERMINATION OF THE PERMUTATION IDENTITIES WHICH ENSURE THAT A SEMIGROUP VARIETY IS FINITELY BASED*

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It is shown that every semigroup variety admitting the permutation identity $x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$ is finitely based if and only if $1\pi \neq 1$ or $n\pi \neq n$.

Introduction and preliminaries

A semigroup variety is *finitely based* if it admits a finite set of identities from which all others are derivable. The classic result in this topic is that all commutative varieties are finitely based [3, Theorem 9]. A natural attempt to generalise this result is to replace 'commutative' by some other class of permutation identities, and indeed Pollak [4, Theorem 3], and Aizenstat [1] have both shown that a semigroup variety which admits a permutation identity of the form $x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$, where $1\pi \neq 1$ and $n\pi \neq n$, is finitely based. Pollak has also observed that this result can be deduced from a theorem of Putcha and Yaqub on consequences of permutation identities [6, Theorem 2]. In this paper we show that we may weaken the condition on the permutation identity to $1\pi \neq 1$ or $n\pi \neq n$. An example of Perkins [3, Theorem 2] shows that our condition is necessary to ensure that a permutation identity guarantee that all varieties admitting it are finitely based, thus determining all permutation identities with this property.

We shall say that a semigroup identity has the *finite basis property* (FBP) if every variety admitting it is finitely based.

Fundamental work on the determination of the identities with FBP has been done by Pollak [5]. He has shown that an identity has FBP only if it has one of twelve possible forms, one of which is the form $x_1x_2\cdots x_n = f(x_1, x_2, \ldots, x_m)$ [5, Theorem 2(a)]. Hence determination of the permutation identities with FBP represents a little progress on this formidable general question.

* The Main Theorem was first obtained by A. Ya. Aizenstat, "On permutative identities", Sovremennaya Algebra 3 (1975) 3–12 (in Russian).

A list of identities defining a variety is called a *presentation* of the variety. Observe that every presentation of a finitely based variety contains a finite presentation. A variety $\mathscr{V}$ is *uniformly periodic* if it admits an identity of the form $x^m = x^{m+n}$ for some $m, n \geq 1$. Observe that a variety is either uniformly periodic or all its identities are consequences of the commutative identity. These identities are called *balanced* as each variable appearing in such an identity occurs equally often on both sides, and we call a variety *balanced* if all its identities are balanced.

Example 1 [3, Theorem 2]. *The variety* $[xyzw = xzyw,\ yx^k y = xyx^{k-2}yx,\ k = 2, 3, \ldots]$ *is not finitely based.*

It follows that a permutation identity has FBP only if it is not a consequence of the *normal identity*, $xyzw = xzyw$, that is only if it has the form $x_1x_2 \cdots x_n = x_{1\pi}x_{2\pi} \cdots x_{n\pi}$, where π is a permutation on $\{1, 2, \ldots, n\}$ and either $1\pi \neq 1$ or $n\pi \neq n$.

Result 2 [3, Theorem 22]. *All uniformly periodic varieties admitting a non-trivial permutation identity are finitely based.*

This result allows us to restrict our attention to balanced varieties, while the next result will let us circumvent the difficulties which arise when dealing with arbitrary permutation identities.

Result 3 [2, Proposition 3.1(ii)]. *Every permutation identity of the form* $x_1x_2 \cdots x_n = x_{1\pi}x_{2\pi} \cdots x_{n\pi}$ *where* $n\pi \neq n$ *implies the identity* $x_1x_2 \cdots x_n xy = x_1x_2 \cdots x_n yx$.

For a word $f = f(x_1, \ldots, x_n)$ we denote the length of f by $|f|$. The *content* of f, denoted by $C(f)$, is the set of variables occurring in f.

The Main Theorem

Theorem 4. *A permutation identity* $x_1x_2 \cdots x_n = x_{1\pi}x_{2\pi} \cdots x_{n\pi}$, *where* π *is a permutation on* $\{1, 2, \ldots, n\}$, *has FBP if and only if either* $1\pi \neq 1$ *or* $n\pi \neq n$.

Proof. The necessity of the condition on the permutation identity was shown in the remark following Example 1.

To prove the converse, suppose to the contrary that there exists a permutation identity $\phi : x_1x_2 \cdots x_n = x_{1\pi}x_{2\pi} \cdots x_{n\pi}$ such that $n\pi \neq n$, which does not have FBP. Then there exists an infinite sequence of balanced identities, $\phi, \phi_1, \phi_2, \ldots$, such that ϕ_k is not a consequence of its predecessors in the sequence, $k = 1, 2, \ldots$. By Result 3, we may replace ϕ by $\phi_0 : x_1x_2 \cdots x_n xy = x_1x_2 \cdots x_n yx$, and the previous statement remains true.

For any balanced identity $\psi: f=g$, we have $|f|=|g|$, so we may unambiguously speak of the length of ψ, denoted by $|\psi|$.

Since there are only finitely many distinct balanced identities of length less than $n+1$, without loss we may assume that $|\phi_i|>n$ for all $i=0,1,2,\ldots$, and so write ϕ_i in the form $\phi_i: p_i f_i = q_i g_i$, where $|p_i|=|q_i|=n$ and $|f_i|=|g_i|>1$, for all $i=1,2,\ldots$.

Let $\bar{\phi}_i$ be an identity of the form $p_i\bar{f}_i=q_i\bar{g}_i$ where $f_i=\bar{f}_i$, $g_i=\bar{g}_i$ are balanced identities (that is the words $p_i\bar{f}_i, q_i\bar{g}_i$ are obtained from $p_i f_i, q_i g_i$ by permuting the variables of f_i and g_i respectively). Observe that $\bar{\phi}_i$ is a consequence of the conjunction of ϕ_0 and $\bar{\phi}_i$. This allows us to replace ϕ_i by $\bar{\phi}_i$ in the list $\phi_0, \phi_1, \ldots$, and still maintain a sequence of balanced identities of length greater than n, no one of which is a consequence of its predecessors. In the course of the proof we shall rearrange the ϕ_i in such a fashion without comment, and we shall call the new identity by the same name, ϕ_i.

There are only finitely many possibilities for $|C(p_i)|, |C(q_i)|$ and $|C(p_i)\cap C(q_i)|$. Therefore, by passing to a suitable subsequence, we may assume that $|C(p_i)|$, $|C(q_i)|$ and $|C(p_i)\cap C(q_i)|$ are independent of i, $i=1,2,\ldots$, and so we may assume that $C(p_i)=\{x_1,x_2,\ldots,x_l\}$, $C(q_i)=\{x_k,x_{k+1},\ldots,x_t\}$ say, with $l>1$, $k<l+1$, for all $i=1,2,\ldots$. Moreover, since there are only finitely many words of length n in a given set of variables, there are only finitely many possibilities for p_i and for q_i. Therefore, again passing to a suitable subsequence, we may assume that p_i and q_i are independent of i, that is without loss we may assume that each ϕ_i, $i>1$, has the form $\phi_i: pf_i=qg_i$ with $|p|=|q|=n$, $|f_i|=|g_i|\geq 1$.

We now write the left hand side of ϕ_i, $i\geq 1$, as

$$px_1^{r_1}x_2^{r_2}\cdots x_t^{r_t}x_{t+1}^{s_1}x_{t+2}^{s_2}\cdots x_{t+m}^{s_m}$$

where $r_j\geq 0$, $s_1\geq s_2\geq\cdots\geq s_m\geq 1$ ($r_1,\ldots,r_t,s_1,\ldots,s_m$ and m all depending on i). We associate a pair of vectors (u_i,v_i) with $\phi_i: u_i=(r_1,r_2,\ldots,r_t)$, $v_i=(s_1,s_2,\ldots,s_m,0,0,\ldots)$, and define a partial order on the $\{\phi_i\}_{i\in\mathbb{N}}$ by $\phi_i\leq\phi_{i'}$ if $u_i\leq u_{i'}$ and $v_i\leq v_{i'}$, where the partial order on the u's and on the v's is that induced by the natural lineal order on entries.

We now observe that if $\phi_i\leq\phi_{i'}$, then the conjunction of ϕ_0 and ϕ_i implies $\phi_{i'}$. To see this we consider the left hand side of $\phi_{i'}$:

$$pf_{i'}=px_1^{r_1'}c_2^{r_2'}\cdots x_t^{r_t'}x_{t+1}^{s_1'}\cdots x_{t+m}^{s_m'}$$

and rewrite it in the form pf_ih where

$$h=x_1^{r_1'-r_1}x_2^{r_2'-r_2}\cdots x_t^{r_t'-r_t}x_{t+1}^{s_1'-s}\cdots x_{t+m'}^{s_m{}'-s_{m'}{}'},$$

$(s_{m+1},s_{m+2},\ldots,s_{m'}=0)$. Applying the identity ϕ_i to pf_ih yields $qg_ih=qg_{i'}$, as ϕ_i and $\phi_{i'}$ are balanced.

Therefore to complete the proof it is enough to show that given a sequence of pairs, $(u_1v_1),(u_2,v_2),\ldots,(u_r,v_r),\ldots$, where the u_i and v_i are defined as before, there exists $i<i'$ such that $u_1\leq u_{i'}$ and $v_i\leq v_{i'}$. Since the u_i's are t-dimensional, it is easy to show by induction on t, that the sequence of u_i's contains an increasing chain.

Therefore without loss we assume that $u_1 \le u_2 \le \cdots \le u_r \le \cdots$. It remains to show that for some $i<i'$, $v_i \le v_{i'}$. This is a special case of the following result in which we call an infinite vector with entries from $\mathbb{N} \cup \{0\}$ a *vector* over $\mathbb{N}^0$. We call a vector over $\mathbb{N}^0$, $(n_1, n_2, \ldots)$, *decreasing* if $n_s \ge n_t$ for all $s<t$ $(s, t \in \mathbb{N})$.

Lemma 5. *If $S_0 = (v_1, v_2, \ldots)$ is a sequence of decreasing vectors over $\mathbb{N}^0$, then there exists $i<i'$ such that $v_i \le v_{i'}$.*

Proof. We first prove the statement of the lemma for the case where each $v_i = (v_{i1}, v_{i2}, \ldots)$ has only finitely many non-zero entries (note that this special case is all that is required to complete the proof of Theorem 4).

Suppose that $v_1 \not\le v_i$ for all $i \ne 1$. Then for each $i \ne 1$ there exists k such that $v_{ik} < v_{1k}$. Since only finitely many entries of v_1 are non-zero, there are only finitely many possibilities for k. This allows us to extract a subsequence S_1 from S_0, where S_1 consists of those v_i such that $v_{ik(1)} < v_{1k(1)}$ for some positive integer $k(1)$.

If possible, we repeat this process and construct successive subsequences $S_0, S_1, S_2, \ldots,$ with S_{j+1} a subsequence of S_j, and list the corresponding integers $k(1), k(2), \ldots$. By deletion of some of the S_j if necessary, we may assume that $k(1) \le k(2) \le \cdots$. Denote $v_{1k(1)}$ by n.

Suppose it is possible to construct $S_0, S_1, S_2, \ldots, S_{n+1}$. For each member of S_1, all entries after the $k(1)$st are less than n. For each member of S_2, all entries after the $k(2)$nd are less than the $k(2)$nd entry in the first member of S_1, which in turn is less than n because $k(2) \ge k(1)$. Hence for each member of S_2, all entries after the $k(2)$nd are less than $n-1$. By repetition of this argument, we derive the contradiction that for each member of S_{n+1}, all entries after the $k(n+1)$th are less than 0. Hence, at some point the above construction fails: that is for some subsequence S_j, beginning with v_i say, there exists $v_{i'}$ in S_j, with $i<i'$, such that $v_i \le v_{i'}$. Since S_j is a subsequence of S_0, this completes the proof in this case.

To complete the proof of the lemma we now return to the general case and for each v_i we define the minimum of v_i, denoted by $\min v_i$, to the least number which occurs in v_i. If we can extract a subsequence from S_0 whose vectors all have the same minimum, the result follows by the preceding argument. If not, we may extract a subsequence $w_1, w_2, \ldots,$ from S_0 such that $\min w_i > \min w_j$ for all $i<j$. Consider the sequence of vectors, $\bar{w}_1, \bar{w}_2, \ldots,$ where $\bar{w}_i$ is obtained from w_i by replacing all entries of w_i that equal $\min w_i$ by 0. By the argument above, there exists $i<i'$ such that $\bar{w}_i \le \bar{w}_{i'} \le \bar{w}_{i'}$. Compare the entries of w_i to those of $w_{i'}$: since w_i agrees with $\bar{w}_i$ for all entries up to the first appearance of $\min w_i$, we have that the entries of w_i up to this point are less than or equal to their counterparts in $w_{i'}$, while all entries of w_i after the first appearance of $\min w_i$ are equal to $\min w_i < \min w_{i'}$ which is less than or equal to all entries of $w_{i'}$, hence $w_i \le w_{i'}$, thus completing the proof of the lemma.

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